

RISING

Education Zone

"Raise your knowledge to raise yourself"...

NEET — JEE MAINS - ADVANCED

Chapter 6 : Systems of Particles and Rotational Motion (Notes)

by - Tomar Sir

Notes

GSEB and CBSE Board Exam preparation subjective notes · Class 11 Physics · NCERT Chapter 6

Q.1 What is a rigid body? Distinguish between pure translation, pure rotation and rolling motion. (2 Marks)

Rigid body: A body with a perfectly definite and unchanging shape. The distances between all pairs of particles of the body do not change, even when forces act on it.

Pure translational motion: Every particle of the body has the **same velocity** at any instant. Example: a rectangular block sliding down an inclined plane without sideways movement (Fig. 6.1) — points P_1 and P_2 move with the same velocity.

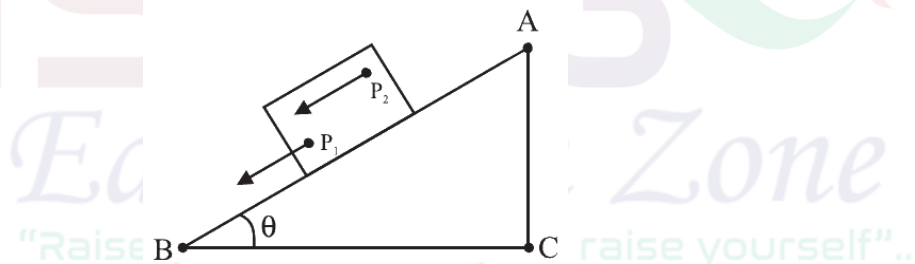


Fig. 6.1: Translational (sliding) motion of a block down an inclined plane. Any point like P_1 or P_2 moves with the same velocity at any instant

Rolling motion: A combination of translation and rotation. For a cylinder rolling down an incline (Fig. 6.2), points P_1, P_2, P_3, P_4 have different velocities; the point of contact P_3 is momentarily at rest if it rolls without slipping.

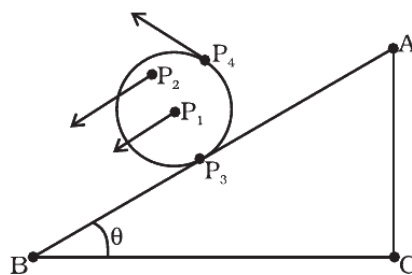


Fig. 6.2: Rolling motion of a cylinder. It is not pure translational motion; the point of contact P_3 has zero velocity at any instant if the cylinder rolls without slipping

Pure rotational motion (about a fixed axis): Every particle of the rigid body moves in a **circle**, which lies in a plane perpendicular to the axis and has its centre on the axis. Examples: a ceiling fan, a potter's wheel, a merry-go-round.

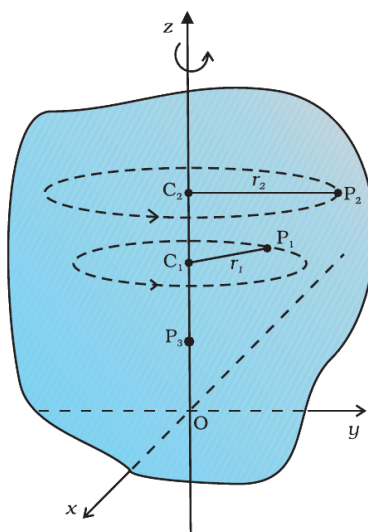


Fig. 6.4: A rigid body rotating about the z-axis. Each point such as P_1 or P_2 describes a circle with its centre on the axis; a point on the axis like P_3 remains stationary

Key Examination Points for Full Marks:

- The motion of a rigid body that is not pivoted or fixed is either pure translation or a combination of translation and rotation.
- A body pivoted at one point (e.g. a spinning top) can have **precession** — the axis itself moves around a cone.

Q.2 Define the centre of mass of a system of particles. Derive its coordinates for two particles and for n particles. (3 Marks)

Centre of mass (CM): The point at which the whole mass of the system may be regarded as concentrated for describing its translational motion.

Two particles: For masses m_1 and m_2 at distances x_1 and x_2 from the origin on the x-axis (Fig. 6.7):

$$X = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

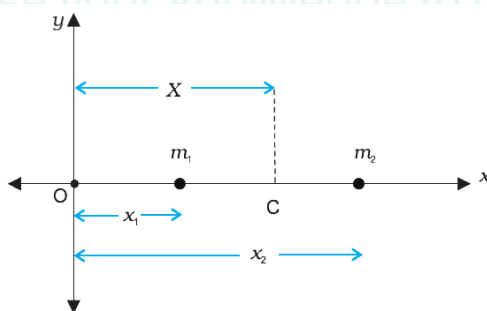


Fig. 6.7

Fig. 6.7: The CM C of two particles m_1 and m_2 on the x-axis

For equal masses, $X = \frac{x_1 + x_2}{2}$ — the CM is midway between them.

n particles (in space):

$$X = \frac{\sum m_i x_i}{M}, \quad Y = \frac{\sum m_i y_i}{M}, \quad Z = \frac{\sum m_i z_i}{M}$$

In vector form:

$$\vec{R} = \frac{\sum m_i \vec{r}_i}{M}, \quad M = \sum m_i$$

Continuous body: $X = \frac{1}{M} \int x \, dm$, and similarly for Y and Z.

Key Examination Points for Full Marks:

- If the origin is at the CM, $\sum m_i \vec{r}_i = 0$.
- For homogeneous bodies with symmetry (rod, ring, disc, sphere, cube), the CM coincides with the **geometric centre**.
- The CM need not lie on the body — e.g. a ring or a hollow sphere.
- **Example 6.1:** masses 100 g, 150 g, 200 g at the corners of an equilateral triangle of side 0.5 m: $X = \frac{5}{18} \text{ m}$, $Y = \frac{1}{3\sqrt{3}} \text{ m}$ — not the geometric centre because the masses are unequal.

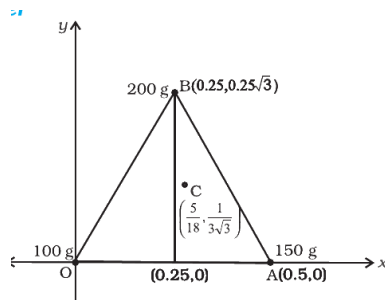


Fig. 6.9

Fig. 6.9: Centre of mass of three masses at the corners of an equilateral triangle (Example 6.1)

Q.3 Find the centre of mass of (a) a triangular lamina (b) an L-shaped lamina. (2 Marks)

(a) Triangular lamina (Example 6.2): Divide the lamina LMN into narrow strips parallel to the base MN (Fig. 6.10). By symmetry, each strip has its CM at its midpoint. Joining these midpoints gives the median LP, so the CM lies on LP. Similarly, it lies on the medians MQ and NR. Hence **the CM is at the centroid G** — the point of concurrence of the medians.

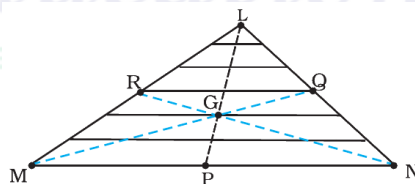


Fig. 6.10

Fig. 6.10: A triangular lamina divided into narrow strips parallel to the base

(b) L-shaped lamina (Example 6.3): The lamina of mass 3 kg consists of three squares of side 1 m, each of mass 1 kg, with centres at $(1/2, 1/2)$, $(3/2, 1/2)$ and $(1/2, 3/2)$ (Fig. 6.11).

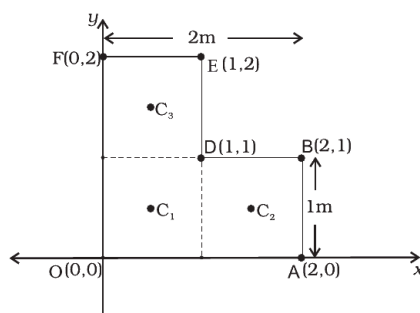


Fig. 6.11

Fig. 6.11: An L-shaped lamina made of three unit squares

$$X = \frac{1(1/2) + 1(3/2) + 1(1/2)}{3} = \frac{5}{6} \text{ m}, \quad Y = \frac{1(1/2) + 1(1/2) + 1(3/2)}{3} = \frac{5}{6} \text{ m}$$

The CM lies on the line OD — the line of symmetry of the L-shape.

Key Examination Points for Full Marks:

- Use symmetry first: the CM always lies on an axis or plane of symmetry.
- For a lamina with a hole cut out, treat the hole as a **negative mass**.

Q.4 Show that the centre of mass of a system moves as if all its mass were concentrated there and all external forces acted on it. (3 Marks)

From $M\vec{R} = \sum m_i \vec{r}_i$, differentiating with respect to time (masses constant):

$$M \frac{d\vec{R}}{dt} = \sum m_i \frac{d\vec{r}_i}{dt} \Rightarrow M\vec{V} = \sum m_i \vec{v}_i$$

Differentiating again:

$$M\vec{A} = \sum m_i \vec{a}_i = \vec{F}_1 + \vec{F}_2 + \dots + \vec{F}_n$$

Each $\vec{F}_i$ includes **external** forces and **internal** forces. By Newton's third law, internal forces occur in equal and opposite pairs and cancel in the sum. Hence

$$M\vec{A} = \vec{F}_{ext}$$

Statement: The centre of mass of a system of particles moves as if all the mass of the system were concentrated at the centre of mass and all the external forces were applied at that point.

Illustration: A projectile that explodes in mid-air (Fig. 6.12) — the explosion forces are internal, so the CM of the fragments continues along the **same parabolic path**.

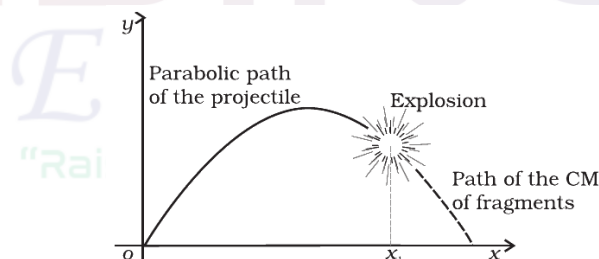


Fig. 6.12: The CM of the fragments of the projectile continues along the same parabolic path

Key Examination Points for Full Marks:

- To find the motion of the CM, no knowledge of internal forces is needed.
- In radioactive decay ($\text{Ra} \rightarrow \text{Rn} + \alpha$), or for a binary star system with no external force, the CM moves uniformly in a straight line.

Q.5 State and prove the law of conservation of linear momentum for a system of particles. (2 Marks)

The total linear momentum of a system is the vector sum of the momenta of all its particles:

$$\vec{P} = \sum m_i \vec{v}_i = M\vec{V}$$

Differentiating: $\frac{d\vec{P}}{dt} = M \frac{d\vec{V}}{dt} = M\vec{A}$. Since $M\vec{A} = \vec{F}_{ext}$,

$$\frac{d\vec{P}}{dt} = \vec{F}_{ext}$$

This is **Newton's second law extended to a system of particles**.

Conservation: If $\vec{F}_{ext} = 0$, then $\frac{d\vec{P}}{dt} = 0$, i.e. $\vec{P} = \text{constant}$.

Statement: When the total external force on a system of particles is zero, the total linear momentum of the system is constant. Equivalently, the **velocity of the CM remains constant**.

In component form: $P_x = c_1, P_y = c_2, P_z = c_3$.

Key Examination Points for Full Marks:

- Individual particles may have complicated trajectories due to internal forces, but the CM moves uniformly (like a free particle).
- In the CM frame, the two products of a decay fly back to back.

Q.6 Define the vector product of two vectors. State its properties. (3 Marks)

Vector product: The vector product of $\vec{a}$ and $\vec{b}$ is a vector $\vec{c} = \vec{a} \times \vec{b}$ such that:

- i. its magnitude is $c = ab \sin \theta$, where θ is the (smaller) angle between $\vec{a}$ and $\vec{b}$;
- ii. $\vec{c}$ is **perpendicular** to the plane containing $\vec{a}$ and $\vec{b}$;
- iii. its direction is given by the **right-handed screw rule** (or right-hand rule): curl the fingers of the right hand from $\vec{a}$ to $\vec{b}$; the stretched thumb points along $\vec{c}$.

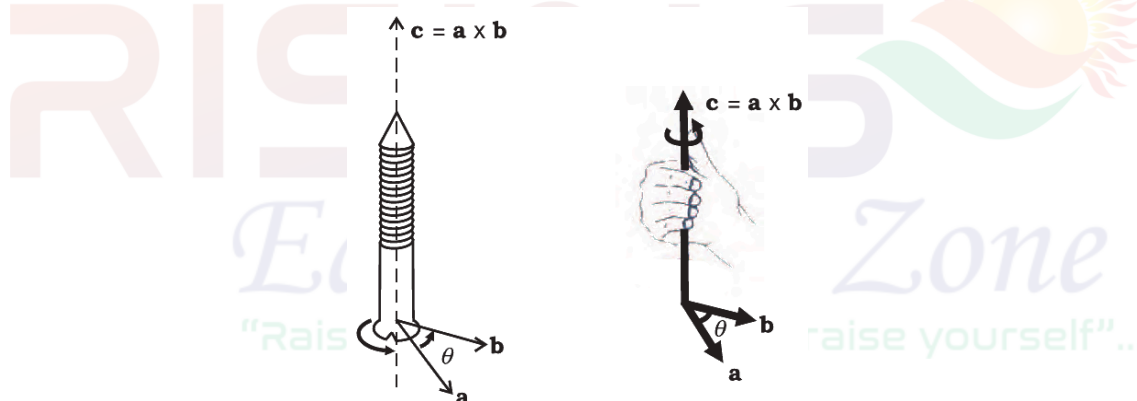


Fig. 6.15: (a) The right-handed screw rule and (b) the right-hand rule for the direction of the vector product

Properties:

- Not commutative:** $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$.
- Distributive:** $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$.
- Under reflection ($x \rightarrow -x$, etc.), $\vec{a} \times \vec{b}$ **does not change sign**.
- $\vec{a} \times \vec{a} = 0$; in particular $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$.
- $\hat{i} \times \hat{j} = \hat{k}, \hat{j} \times \hat{k} = \hat{i}, \hat{k} \times \hat{i} = \hat{j}$ (cyclic order positive); $\hat{j} \times \hat{i} = -\hat{k}$, etc.

Component (determinant) form:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = (a_y b_z - a_z b_y) \hat{i} + (a_z b_x - a_x b_z) \hat{j} + (a_x b_y - a_y b_x) \hat{k}$$

Key Examination Points for Full Marks:

- **Example 6.4:** $\vec{a} = 3\hat{i} - 4\hat{j} + 5\hat{k}$, $\vec{b} = -2\hat{i} + \hat{j} - 3\hat{k}$; $\vec{a} \cdot \vec{b} = -6 - 4 - 15 = -25$; $\vec{a} \times \vec{b} = 7\hat{i} - \hat{j} - 5\hat{k}$;
 $\vec{b} \times \vec{a} = -7\hat{i} + \hat{j} + 5\hat{k}$.
- Area of a parallelogram $= |\vec{a} \times \vec{b}|$; area of a triangle $= \frac{1}{2} |\vec{a} \times \vec{b}|$.

Q.7 Define angular velocity and angular acceleration. Derive $\vec{v} = \vec{\omega} \times \vec{r}$. (2 Marks)

In rotation about a fixed axis, a particle P moves in a circle with centre C on the axis (Fig. 6.16). If it turns through $\Delta\theta$ in time Δt , the **angular velocity** is

$$\omega = \lim_{\Delta t \rightarrow 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

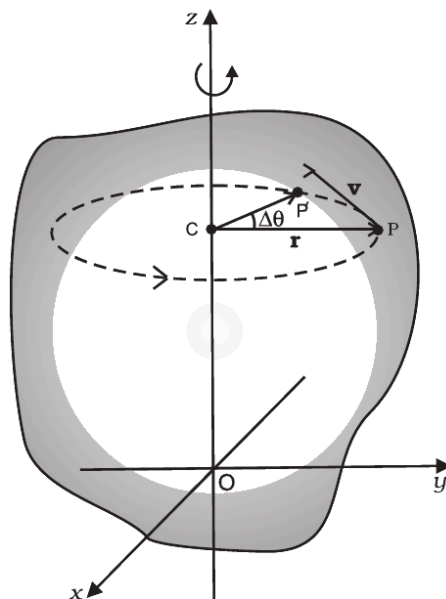


Fig. 6.16: Rotation about a fixed axis. A particle P of the rigid body moves in a circle with centre C on the axis

The linear speed of a particle at perpendicular distance $r_{\perp}$ from the axis is $v = \omega r_{\perp}$. All particles of a rigid body have the **same** ω at an instant — so ω is the angular velocity of the whole body. Particles on the axis have $v = 0$.

Angular velocity is a vector, directed **along the axis** of rotation, in the direction in which a right-handed screw would advance if turned with the body.

Relation $\vec{v} = \vec{\omega} \times \vec{r}$: Let $\vec{r} = \vec{OP} = \vec{OC} + \vec{CP}$, with O on the axis. Since $\vec{\omega}$ is along OC, $\vec{\omega} \times \vec{OC} = 0$. So

$$\vec{\omega} \times \vec{r} = \vec{\omega} \times \vec{CP}$$

This vector is perpendicular to both $\vec{\omega}$ and CP — i.e. **tangent** to the circle — with magnitude $\omega r_{\perp}$. This is exactly the linear velocity:

$$\vec{v} = \vec{\omega} \times \vec{r}$$

Angular acceleration: $\vec{\alpha} = \frac{d\vec{\omega}}{dt}$. For a fixed axis, it reduces to the scalar $\alpha = \frac{d\omega}{dt}$.

Key Examination Points for Full Marks:

- Units: ω in rad s^{-1} [T^{-1}]; α in rad s^{-2} [T^{-2}].
- For a fixed axis, the direction of $\vec{\omega}$ does not change; only its magnitude may change.

Q.8 Define torque (moment of force). State its unit, dimensions and when it is zero. (2 Marks)

Torque: The rotational analogue of force. For a force $\vec{F}$ acting on a particle at position $\vec{r}$ relative to the origin O (Fig. 6.18):

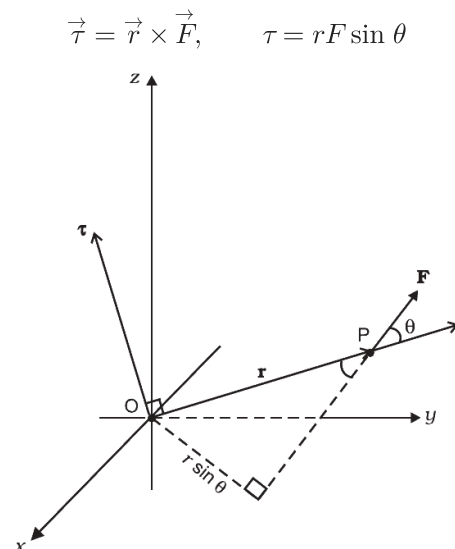


Fig. 6.18: $\tau = r \times F$; τ is perpendicular to the plane containing r and F , with direction given by the right-handed screw rule

The magnitude can be written as

$$\tau = (r \sin \theta) F = r_{\perp} F \quad \text{or} \quad \tau = r (F \sin \theta) = r F_{\perp}$$

where $r_{\perp}$ is the perpendicular distance of the line of action of $\vec{F}$ from O (the **moment arm**).

Unit: N m. **Dimensions:** $[ML^2T^{-2}]$ — same as work, but torque is a **vector** and work is a scalar.

Torque is zero if $r = 0$, $F = 0$, or $\theta = 0^\circ$ or 180° (the line of action passes through the origin).

Key Examination Points for Full Marks:

- Reversing $\vec{F}$ reverses $\vec{\tau}$; reversing both $\vec{r}$ and $\vec{F}$ leaves $\vec{\tau}$ unchanged.
- Example 6.5:** $\vec{F} = 7\hat{i} + 3\hat{j} - 5\hat{k}$ at $\vec{r} = \hat{i} - \hat{j} + \hat{k}$: $\vec{\tau} = 2\hat{i} + 12\hat{j} + 10\hat{k}$.

Q.9 Define angular momentum of a particle. Show that $d\vec{l}/dt = \vec{\tau}$. (3 Marks)

Angular momentum of a particle of momentum $\vec{p}$ at position $\vec{r}$ about the origin O:

$$\vec{l} = \vec{r} \times \vec{p}, \quad l = rp \sin \theta = r_{\perp} p = rp_{\perp}$$

Unit: $\text{kg m}^2 \text{s}^{-1}$ ($= \text{J s}$). Dimensions: $[ML^2T^{-1}]$. It is zero if $p = 0$, $r = 0$, or the line of $\vec{p}$ passes through O.

Relation with torque: Differentiate $\vec{l} = \vec{r} \times \vec{p}$:

$$\frac{d\vec{l}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

The first term is $\vec{v} \times m\vec{v} = 0$ (parallel vectors). Since $\frac{d\vec{p}}{dt} = \vec{F}$:

$$\frac{d\vec{l}}{dt} = \vec{r} \times \vec{F} = \vec{\tau}$$

The time rate of change of angular momentum of a particle equals the torque acting on it — the rotational analogue of $\vec{F} = d\vec{p}/dt$.

Key Examination Points for Full Marks:

- **Example 6.6:** A particle moving with constant velocity has constant angular momentum about any point: $l = mvr \sin \theta = mv (OM)$, and the perpendicular distance OM is constant; the direction (into the page) is also constant. No external torque acts.

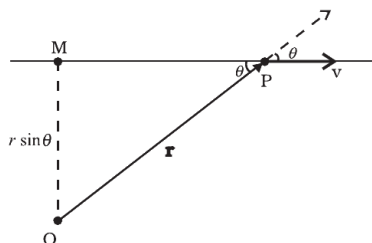


Fig 6.19

Fig. 6.19: A particle moving with constant velocity v ; its moment arm $OM = r \sin \theta$ about O is constant

Q.10 Derive $dL/dt = \tau_{ext}$ for a system of particles and state the law of conservation of angular momentum. (3 Marks)

For a system of n particles, $\vec{L} = \sum \vec{l}_i = \sum \vec{r}_i \times \vec{p}_i$. Then

$$\frac{d\vec{L}}{dt} = \sum \frac{d\vec{l}_i}{dt} = \sum \vec{\tau}_i$$

Each torque $\vec{\tau}_i = \vec{r}_i \times \vec{F}_i$ comes from external and internal forces: $\vec{\tau} = \vec{\tau}_{ext} + \vec{\tau}_{int}$. If the internal forces between any two particles are equal and opposite **and act along the line joining them**, the torque of each pair is zero, so $\vec{\tau}_{int} = 0$. Hence

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{ext}$$

Conservation of angular momentum: If $\vec{\tau}_{ext} = 0$, then $\frac{d\vec{L}}{dt} = 0$, i.e. $\vec{L} = \text{constant}$. Equivalently, $L_x = K_1$, $L_y = K_2$, $L_z = K_3$.

Statement: If the total external torque on a system of particles is zero, the total angular momentum of the system is conserved.

Key Examination Points for Full Marks:

- This holds for any system — rigid or with internal motion.
- It is the rotational analogue of the conservation of linear momentum.

Q.11 State the conditions for equilibrium of a rigid body. What is a couple? (3 Marks)

A rigid body is in **mechanical equilibrium** if both its linear momentum and angular momentum are not changing with time:

(1) **Translational equilibrium:** $\sum \vec{F}_i = 0$, i.e. $\sum F_{ix} = \sum F_{iy} = \sum F_{iz} = 0$.

(2) **Rotational equilibrium:** $\sum \vec{\tau}_i = 0$, i.e. $\sum \tau_{ix} = \sum \tau_{iy} = \sum \tau_{iz} = 0$.

These are **six independent conditions**. For coplanar forces, only **three** are needed: two force components and the torque about an axis perpendicular to the plane. If the translational condition holds, the rotational condition is independent of the origin.

Partial equilibrium: In Fig. 6.20(a), two equal parallel forces F act in the same direction at the ends of a light rod — the torques cancel (rotational equilibrium) but $\sum \vec{F} \neq 0$. In Fig. 6.20(b), the forces are equal and opposite — the body is in translational equilibrium but rotates.

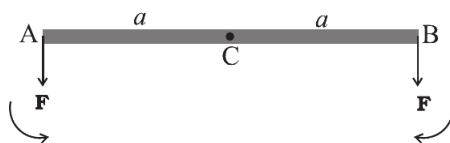


Fig. 6.20 (a)

Fig. 6.20(a): Equal parallel forces in the same direction: rotational, but not translational, equilibrium

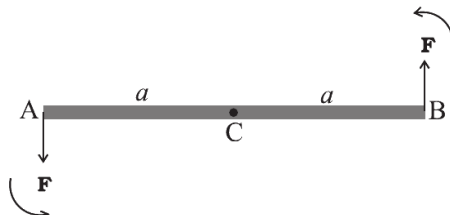


Fig. 6.20(b): Equal and opposite forces with different lines of action — a couple: translational, but not rotational, equilibrium

Couple: A pair of forces of equal magnitude acting in opposite directions with **different lines of action**. It produces rotation without translation. Examples: opening the lid of a bottle; the earth's magnetic field acting on a compass needle.

Moment of a couple is independent of the origin (Example 6.7): With $-\vec{F}$ at A ($\vec{r}_1$) and $\vec{F}$ at B ($\vec{r}_2$):

$$\vec{r}_1 \times (-\vec{F}) + \vec{r}_2 \times \vec{F} = (\vec{r}_2 - \vec{r}_1) \times \vec{F} = \vec{AB} \times \vec{F}$$

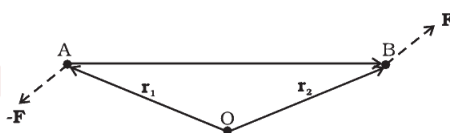


Fig. 6.22

Fig. 6.22: A couple: forces F and $-F$ acting at B and A

Q.12 State the principle of moments. Define mechanical advantage and centre of gravity. (2 Marks)

Principle of moments (lever): For an ideal lever pivoted at the fulcrum O (Fig. 6.23), with load F_1 at distance d_1 and effort F_2 at d_2 :

- Translational equilibrium: $R - F_1 - F_2 = 0$
- Rotational equilibrium (moments about O): $d_1 F_1 - d_2 F_2 = 0$

$$d_1 F_1 = d_2 F_2 \quad (\text{load arm} \times \text{load} = \text{effort arm} \times \text{effort})$$

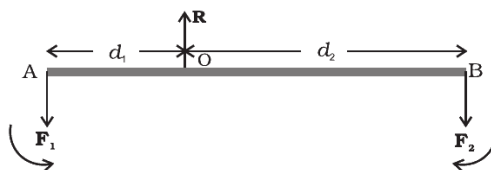


Fig. 6.23

Fig. 6.23: A lever with fulcrum O; F_1 is the load and F_2 the effort

Mechanical advantage:

$$\text{M.A.} = \frac{F_1}{F_2} = \frac{d_2}{d_1}$$

If the effort arm is longer than the load arm, $\text{M.A.} > 1$ — a small effort can lift a large load. Examples: see-saw, beam balance.

Centre of gravity (CG): The point where the total gravitational torque on the body is zero:

$$\vec{\tau}_g = \sum \vec{r}_i \times m_i \vec{g} = 0$$

Since $\vec{g}$ is the same for all particles, $\sum m_i \vec{r}_i = 0$, so **the CG coincides with the CM** in a uniform gravitational field. For a very large body (where g varies), they do not coincide.

Key Examination Points for Full Marks:

- The CG of an irregular lamina is found by suspending it from two or three points and marking the verticals; they intersect at the CG.
- Example 6.8:** A 70 cm, 4 kg bar on knife-edges 10 cm from each end with a 6 kg load at 30 cm: $R_1 = 54.88$ N, $R_2 = 43.12$ N.

Q.13 Define moment of inertia and radius of gyration. Derive $K = \frac{1}{2}I\omega^2$. (3 Marks)

For a body rotating about a fixed axis with angular velocity ω , a particle at distance r_i from the axis has $v_i = r_i\omega$ and kinetic energy $\frac{1}{2}m_i r_i^2 \omega^2$. The total KE is

$$K = \frac{1}{2} \left(\sum m_i r_i^2 \right) \omega^2$$

Moment of inertia:

$$I = \sum m_i r_i^2 \Rightarrow K = \frac{1}{2} I \omega^2$$

Compare $K = \frac{1}{2}mv^2$: **I is the rotational analogue of mass** (rotational inertia). It depends on the mass, its distribution about the axis, and the position and orientation of the axis — not on ω .

Unit: kg m^2 ; **dimensions:** $[ML^2]$.

Radius of gyration (k): $I = Mk^2$. It is the distance from the axis of a point mass, equal to the body's mass, whose moment of inertia equals that of the body. For a rod about its midpoint, $k = L/\sqrt{12}$; for a disc about its diameter, $k = R/2$.

Simple cases:

- Thin ring rotating about its axis: every element is at distance R , so $I = MR^2$.
- Light rod of length l with masses $M/2$ at each end (Fig. 6.28): $I = 2 \cdot \frac{M}{2} \left(\frac{l}{2} \right)^2 = \frac{Ml^2}{4}$.

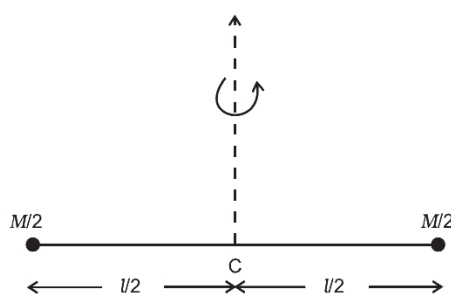


Fig. 6.28: A light rod with a pair of masses rotating about an axis through the CM perpendicular to the rod

Table 6.1: Moments of inertia of regular bodies

Body	Axis	I
Thin ring, radius R	Perpendicular to plane, at centre	MR^2
Thin ring, radius R	Diameter	$MR^2/2$
Thin rod, length L	Perpendicular to rod, at midpoint	$ML^2/12$
Circular disc, radius R	Perpendicular to disc, at centre	$MR^2/2$
Circular disc, radius R	Diameter	$MR^2/4$
Hollow cylinder, radius R	Axis of cylinder	MR^2
Solid cylinder, radius R	Axis of cylinder	$MR^2/2$
Solid sphere, radius R	Diameter	$2MR^2/5$

Key Examination Points for Full Marks:

- A **flywheel** has a large moment of inertia; it resists sudden changes in speed and gives a smooth ride.
- Parallel axis theorem:** $I = I_{cm} + Md^2$. **Perpendicular axis theorem** (lamina): $I_z = I_x + I_y$.

Q.14 Write the equations of rotational kinematics. Derive $\omega = \omega_0 + \alpha t$. (2 Marks)

For rotation about a fixed axis with **uniform angular acceleration**, the equations are analogous to those of linear motion:

Linear	Rotational
$v = v_0 + at$	$\omega = \omega_0 + \alpha t$
$x = x_0 + v_0 t + \frac{1}{2}at^2$	$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = v_0^2 + 2ax$	$\omega^2 = \omega_0^2 + 2\alpha (\theta - \theta_0)$

Derivation (Example 6.10): $\frac{d\omega}{dt} = \alpha = \text{constant}$. Integrating:

$$\omega = \int \alpha dt + c = \alpha t + c$$

At $t = 0$, $\omega = \omega_0$, so $c = \omega_0$. Hence $\omega = \omega_0 + \alpha t$. Integrating $\omega = d\theta/dt$ gives the second equation.

Key Examination Points for Full Marks:

- Example 6.11:** Motor speed rises from 1200 rpm (40π rad/s) to 3120 rpm (104π rad/s) in 16 s: $\alpha = 4\pi$ rad/s²; $\theta = 1152\pi$ rad = **576 revolutions**.
- Convert rpm to rad/s: multiply by $2\pi/60$.

**Q.15 Derive the work done by a torque and power in rotational motion. (3 Marks)
Hence obtain $\tau = I\alpha$.**

Let a force $\vec{F}_1$ act on a particle at P_1 of a body rotating about a fixed axis (Fig. 6.30). In a small rotation $d\theta$, the particle moves $ds_1 = r_1 d\theta$ along the circle. The work done is

$$dW_1 = F_1 ds_1 \cos \phi_1 = F_1 (r_1 d\theta) \sin \alpha_1 = \tau_1 d\theta$$

since $\phi_1 + \alpha_1 = 90^\circ$ and $\tau_1 = r_1 F_1 \sin \alpha_1$. For all forces:

$$dW = \tau d\theta$$

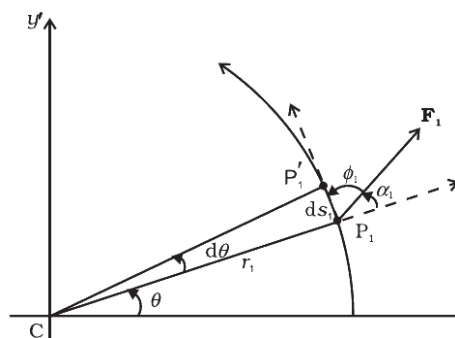


Fig. 6.30: Work done by a force F_1 acting on a particle of a body rotating about a fixed axis

Power:

$$P = \frac{dW}{dt} = \tau \frac{d\theta}{dt} \Rightarrow P = \tau\omega$$

$\tau = I\alpha$: For a rigid body there is no internal motion, so the work done by external torques increases its kinetic energy:

$$\frac{d}{dt} \left(\frac{1}{2} I \omega^2 \right) = I \omega \frac{d\omega}{dt} = I \omega \alpha$$

Equating with $\tau\omega$:

$$\tau = I\alpha$$

This is **Newton's second law for rotation** about a fixed axis — the analogue of $F = ma$.

Table 6.2: Translational vs rotational motion

Linear motion	Rotation about a fixed axis
Displacement x	Angular displacement θ
Velocity $v = dx/dt$	Angular velocity $\omega = d\theta/dt$
Acceleration $a = dv/dt$	Angular acceleration $\alpha = d\omega/dt$
Mass M	Moment of inertia I
Force $F = Ma$	Torque $\tau = I\alpha$
Work $dW = F ds$	Work $dW = \tau d\theta$
Kinetic energy $K = Mv^2/2$	Kinetic energy $K = I\omega^2/2$
Power $P = Fv$	Power $P = \tau\omega$
Linear momentum $p = Mv$	Angular momentum $L = I\omega$

Q.16 Derive $L = I\omega$ for rotation about a fixed axis and explain the conservation of angular momentum with examples.

(3 Marks)

For a particle of a rigid body rotating about the z -axis, with $\vec{r} = \vec{OC} + \vec{CP}$:

$$\vec{l} = \vec{OC} \times m\vec{v} + \vec{CP} \times m\vec{v}$$

The second term is along the axis: $\vec{l}_z = mr_{\perp}^2 \omega \hat{k}$. Summing over all particles:

$$\vec{L}_z = \left(\sum m_i r_i^2 \right) \omega \hat{k} = I \omega \hat{k}$$

For bodies **symmetric about the axis**, the perpendicular components cancel in pairs, so

$$\vec{L} = I\vec{\omega}$$

Since $\frac{d\vec{L}}{dt} = \vec{\tau}$: $\frac{d}{dt} (I\omega) = \tau$. If $\tau_{ext} = 0$:

$$I\omega = \text{constant} \quad (I_1\omega_1 = I_2\omega_2)$$

Examples:

- A girl on a **swivel chair** stretches her arms — I increases, ω decreases; she folds them — ω increases (Fig. 6.32(a)).
- An **acrobat or diver** curls the body to reduce I and spin faster (Fig. 6.32(b)).
- Skaters and dancers performing a pirouette.



Fig. 6.32: (a) Conservation of angular momentum on a swivel chair. (b) An acrobat using the principle

Key Examination Points for Full Marks:

- Rotational KE changes when I changes: $K = \frac{L^2}{2I}$ — pulling arms in **increases** KE (work is done by the muscles).
- **Example 6.12:** A cord pulled with 25 N on a flywheel (20 kg, $R = 20$ cm): $\tau = 5$ N m, $I = 0.4$ kg m², $\alpha = 12.5$ s⁻²; work for 2 m = 50 J = KE gained.

Rising Education Zone
"Raise your knowledge to raise yourself"...

Formula Sheet — Systems of Particles and Rotational Motion

Quantity	Formula	SI Unit
Centre of mass	$\vec{R} = \frac{\sum m_i \vec{r}_i}{M}; X = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$	m
Velocity of CM	$\vec{V} = \frac{\sum m_i \vec{v}_i}{M} = \frac{\vec{P}}{M}$	m s ⁻¹
Motion of CM	$M\vec{A} = \vec{F}_{ext}; \frac{d\vec{P}}{dt} = \vec{F}_{ext}$	N
Vector product	$ \vec{a} \times \vec{b} = ab \sin \theta$	—
Linear velocity	$\vec{v} = \vec{\omega} \times \vec{r}; v = \omega r$	m s ⁻¹
Angular acceleration	$\alpha = \frac{d\omega}{dt}$	rad s ⁻²
Torque	$\vec{\tau} = \vec{r} \times \vec{F}; \tau = rF \sin \theta$	N m
Angular momentum	$\vec{l} = \vec{r} \times \vec{p}; L = I\omega$	J s
Torque–angular momentum	$\frac{d\vec{L}}{dt} = \vec{\tau}_{ext}$	N m
Equilibrium	$\sum \vec{F} = 0, \sum \vec{\tau} = 0$	—
Principle of moments	$d_1 F_1 = d_2 F_2; \text{M.A.} = \frac{d_2}{d_1}$	—
Moment of inertia	$I = \sum m_i r_i^2 = Mk^2$	kg m ²
Rotational KE	$K = \frac{1}{2} I \omega^2 = \frac{L^2}{2I}$	J
Kinematics	$\omega = \omega_0 + \alpha t; \theta = \omega_0 t + \frac{1}{2} \alpha t^2;$ $\omega^2 = \omega_0^2 + 2\alpha\theta$	—
Newton's 2nd law (rotation)	$\tau = I\alpha$	N m
Work and power	$W = \int \tau d\theta; P = \tau\omega$	J, W
Conservation	$I_1 \omega_1 = I_2 \omega_2$	—
Parallel axis theorem	$I = I_{cm} + Md^2$	kg m ²
Perpendicular axis theorem	$I_z = I_x + I_y$ (lamina)	kg m ²
Rolling KE	$K = \frac{1}{2} m v^2 \left(1 + \frac{k^2}{R^2} \right)$	J

Dimensions: angular velocity [T⁻¹]; angular momentum [M L² T⁻¹]; torque [M L² T⁻²]; moment of inertia [M L²].

NEET / JEE Quick Points and Tricks

- CM of two bodies divides the line joining them inversely as the masses:** $r_1 : r_2 = m_2 : m_1$. If two bodies attract each other from rest, they meet at their CM.
- Cut-out (hole) problems:** treat the hole as negative mass. A disc of radius R with a hole of radius R/2 touching the rim: CM shifts R/6 away from the hole; $I = \frac{13}{32} MR^2$ about the original centre.
- No external force** → CM velocity is constant. An explosion, decay or internal force does not change the CM path.

4. **Radius of gyration:** ring (axis) R ; disc (axis) $R/\sqrt{2}$; disc (diameter) $R/2$; solid sphere $R\sqrt{2/5}$; hollow sphere $R\sqrt{2/3}$; rod (centre) $L/\sqrt{12}$; rod (end) $L/\sqrt{3}$.
5. **Tangent axes:** ring, tangent in plane $\frac{3}{2}MR^2$, tangent perpendicular to plane $2MR^2$; disc, tangent in plane $\frac{5}{4}MR^2$, perpendicular $\frac{3}{2}MR^2$; solid sphere tangent $\frac{7}{5}MR^2$.
6. **Rod bent into any shape** about the bending point: just add each part's MI about its end, $\frac{1}{3}mL^2$.
7. **Rolling down an incline:** $a = \frac{g \sin \theta}{1 + k^2/R^2}$, $v = \sqrt{\frac{2gh}{1 + k^2/R^2}}$. Order of arrival: solid sphere > disc/solid cylinder > hollow sphere > ring. Mass and radius do not matter.
8. **Fraction of KE in rolling:** rotational/total = $\frac{k^2}{k^2 + R^2}$: ring 1/2, disc 1/3, solid sphere 2/7.
9. **K, L and I:** $K = \frac{L^2}{2I}$ and $L = \sqrt{2IK}$. Same L – smaller I has more K; same K – larger I has more L.
10. **Earth's radius changes** (mass constant): $T \propto R^2$. If R halves, the day becomes 6 h; if R doubles, 96 h.
11. **Torque from a hinged rod** released horizontally: $\alpha = \frac{3g}{2L}$; the end's initial linear acceleration is $\frac{3}{2}g$ (greater than g).
12. **String on pulley with hanging mass m (pulley = disc of mass M):** $a = \frac{mg}{m + M/2}$.
13. **Angular momentum about a point** for a particle moving in a straight line with constant velocity is constant: $L = mvd$ (d = perpendicular distance). If the line passes through the point, L = 0.
14. **Uniform circular motion:** total acceleration of a rim point = $\sqrt{(\alpha r)^2 + (\omega^2 r)^2}$. Linear velocity of the contact point of a body rolling without slipping is zero; the topmost point moves at $2v$.

