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NEET — JEE MAINS - ADVANCED

Chapter 7 : Gravitation (Notes)

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Notes

GSEB and CBSE Board Exam preparation subjective notes · Class 11 Physics · NCERT Chapter 7

Q.1 State Kepler's three laws of planetary motion.

(3 Marks)

Johannes Kepler (1571–1630) analysed the astronomical data collected by Tycho Brahe and stated three laws.

1. Law of orbits: All planets move in elliptical orbits with the Sun situated at one of the foci of the ellipse.

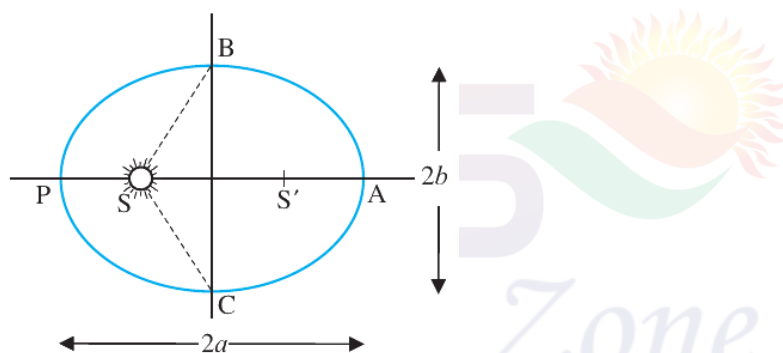


Fig. 7.1(a): An ellipse traced by a planet around the sun. P is the perihelion (closest point) and A the aphelion (farthest point); the semi-major axis is half the distance AP

An ellipse can be drawn by fixing the ends of a string at two points F_1 and F_2 (the **foci**) and moving a pencil keeping the string taut (Fig. 7.1(b)). For any point T on the ellipse, $TF_1 + TF_2$ is constant. A circle is a special case in which the foci merge.

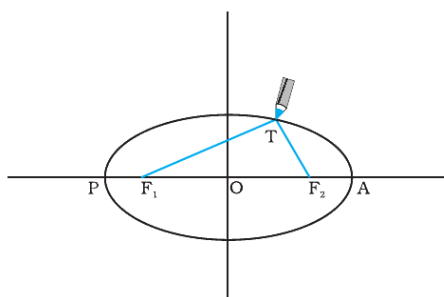


Fig. 7.1(b): Drawing an ellipse using a string with its ends fixed at F_1 and F_2

2. Law of areas: The line that joins any planet to the sun sweeps equal areas in equal intervals of time. Planets move **faster** when nearer the sun and **slower** when farther away.

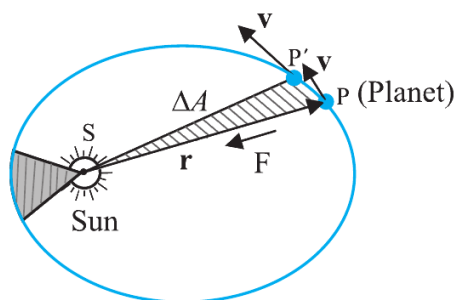


Fig. 7.2: The planet P moves around the sun in an elliptical orbit. The shaded area ΔA is swept in a small interval of time Δt

3. Law of periods: The square of the time period of revolution of a planet is proportional to the cube of the semi-major axis of its elliptical orbit.

$$T^2 \propto a^3 \quad \text{i.e.} \quad \frac{T^2}{a^3} = \text{constant}$$

Key Examination Points for Full Marks:

- Table 7.1 of the NCERT book shows that $Q = T^2/a^3$ is nearly the same ($\approx 3 \times 10^{-34} \text{ y}^2 \text{ m}^{-3}$) for all planets.
- Example 7.1:** At perihelion and aphelion, $mr_p v_p = mr_A v_A \Rightarrow \frac{v_p}{v_A} = \frac{r_A}{r_p}$; since $r_A > r_p$, $v_p > v_A$. The planet takes longer to traverse BAC (near aphelion) than CPB.

Q.2 Show that Kepler's second law follows from the conservation of angular momentum. (2 Marks)

Let the sun be at the origin, and let the position and momentum of the planet be $\vec{r}$ and $\vec{p}$. In a small time Δt , the planet sweeps an area (Fig. 7.2)

$$\Delta \vec{A} = \frac{1}{2} (\vec{r} \times \vec{v} \Delta t)$$

$$\frac{\Delta \vec{A}}{\Delta t} = \frac{1}{2} \frac{(\vec{r} \times \vec{p})}{m} = \frac{\vec{L}}{2m}$$

where $\vec{L}$ is the angular momentum.

Gravity is a **central force** (directed along $\vec{r}$), so its torque $\vec{r} \times \vec{F} = 0$. Hence $\vec{L}$ is constant, and

$$\frac{\Delta A}{\Delta t} = \frac{L}{2m} = \text{constant}$$

This is the **law of areas**.

Key Examination Points for Full Marks:

- The law of areas holds for **any** central force, not only the inverse-square law.
- Since $\vec{L}$ is constant in direction, the orbit lies in a fixed plane.

Q.3 State Newton's universal law of gravitation. Write it in vector form. (3 Marks)

Newton's reasoning: The moon (orbit radius R_m , period T) has a centripetal acceleration

$$a_m = \frac{V^2}{R_m} = \frac{4\pi^2 R_m}{T^2}$$

With $T \approx 27.3$ days and $R_m \approx 3.84 \times 10^8 \text{ m}$, a_m is much smaller than g . If gravity decreases as the inverse square of distance: $\frac{g}{a_m} = \frac{R_m^2}{R_E^2} \approx 3600$, in agreement with the measured values.

Universal law of gravitation: Every body in the universe attracts every other body with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

$$|\vec{F}| = G \frac{m_1 m_2}{r^2}$$

Vector form: The force on m_2 due to m_1 is

$$\vec{F} = -G \frac{m_1 m_2}{r^2} \hat{r} = -G \frac{m_1 m_2}{|\vec{r}|^3} \vec{r}$$

where $\vec{r} = \vec{r}_2 - \vec{r}_1$ and $\hat{r}$ is the unit vector from m_1 to m_2 (Fig. 7.3). The negative sign shows the force is **attractive**. By Newton's third law, $\vec{F}_{12} = -\vec{F}_{21}$.

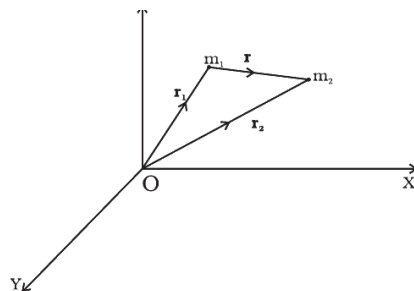


Fig. 7.3: Gravitational force on m_1 due to m_2 is along r , where $r = r_2 - r_1$

Superposition: For several point masses, the force on m_1 is the vector sum (Fig. 7.4):

$$\vec{F}_1 = \frac{Gm_2m_1}{r_{21}^2} \hat{r}_{21} + \frac{Gm_3m_1}{r_{31}^2} \hat{r}_{31} + \frac{Gm_4m_1}{r_{41}^2} \hat{r}_{41}$$

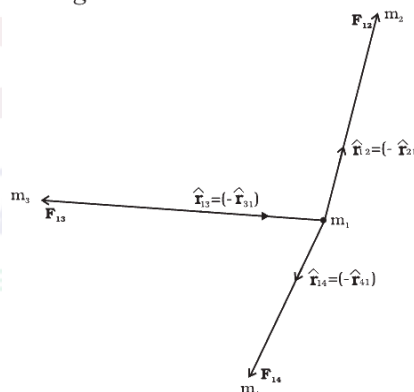


Fig. 7.4: The gravitational force on m_1 is the vector sum of the forces exerted by m_2 , m_3 and m_4

Key Examination Points for Full Marks:

- Example 7.2:** Three equal masses m at the vertices of an equilateral triangle; a mass $2m$ at the centroid ($AG = BG = CG = 1$ m) experiences **zero** net force (by symmetry). If the mass at A is doubled, the net force is $2Gm^2\hat{j}$.

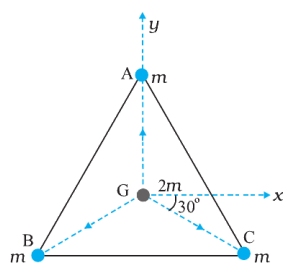


Fig. 7.5: Three equal masses at the vertices of $\triangle ABC$ and a mass $2m$ at the centroid G

Q.4 State the two shell theorems. What is the gravitational constant G (3 Marks) and how was it measured?

Shell theorems:

- 1. The force of attraction between a hollow spherical shell of uniform density and a point mass situated **outside** it is just as if the entire mass of the shell were concentrated at its centre.
- 2. The force of attraction due to a hollow spherical shell of uniform density on a point mass situated **inside** it is zero.

Gravitational constant G : The value of G was first determined experimentally by Henry **Cavendish** in 1798.

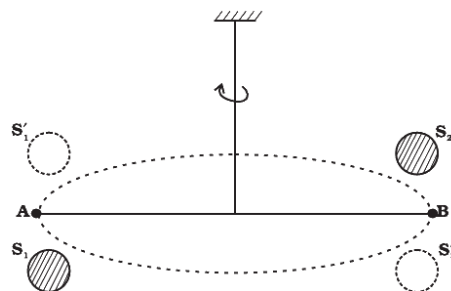


Fig. 7.6: Schematic drawing of Cavendish's experiment. S_1 and S_2 are large spheres kept on either side of the small masses at A and B

Cavendish's experiment: A bar AB with two small lead spheres at its ends is suspended by a fine wire. Two large lead spheres are brought close to the small ones on opposite sides. They attract the small spheres with equal and opposite forces, producing a **torque** $F \times L$ (L = length of the bar) but no net force. The wire twists through angle θ until the restoring torque $\tau\theta$ balances it:

$$G \frac{Mm}{d^2} L = \tau\theta$$

Measuring θ (and τ separately) gives G . The accepted value is

$$G = 6.67 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$$

Key Examination Points for Full Marks:

- Dimensions of G : $[M^{-1} L^3 T^{-2}]$. G is a **universal** constant; g is not.
- Gravitational shielding is **not** possible.

Q.5 Derive the expression for acceleration due to gravity on the earth's (2 Marks) surface. Why is it said "Cavendish weighed the earth"?

Treat the earth as made of concentric spherical shells. A point outside the earth is outside all the shells, so by the shell theorem the earth attracts it as if its entire mass M_E were at the centre. On the surface ($r = R_E$):

$$F = G \frac{M_E m}{R_E^2}$$

By Newton's second law, $F = mg$. Hence

$$g = \frac{F}{m} = \frac{GM_E}{R_E^2}$$

"Cavendish weighed the earth": g is easily measured and R_E is known. Once G was measured by Cavendish, the mass of the earth could be found:

$$M_E = \frac{gR_E^2}{G} = \frac{9.81 \times (6.37 \times 10^6)^2}{6.67 \times 10^{-11}} \approx 5.97 \times 10^{24} \text{ kg}$$

Key Examination Points for Full Marks:

- g does not depend on the mass of the falling body.
- For a planet of the same density ρ : $g = \frac{4}{3}\pi G\rho R$, so $g \propto R$.

Q.6 Derive the variation of g with height above the earth's surface. (3 Marks)

Consider a point mass m at height h above the earth's surface (Fig. 7.8(a)). Its distance from the centre is $(R_E + h)$. Since the point is outside the earth:

$$F(h) = \frac{GM_E m}{(R_E + h)^2}$$

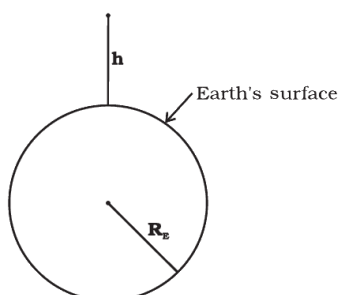


Fig. 7.8(a): g at a height h above the surface of the earth

$$g(h) = \frac{GM_E}{(R_E + h)^2}$$

This is less than the surface value $g = GM_E/R_E^2$. Writing

$$g(h) = \frac{GM_E}{R_E^2 (1 + h/R_E)^2} = g \left(1 + \frac{h}{R_E}\right)^{-2}$$

For $h \ll R_E$, using the binomial expansion:

$$g(h) \simeq g \left(1 - \frac{2h}{R_E}\right)$$

Key Examination Points for Full Marks:

- For small heights, g decreases by a fraction $2h/R_E$.
- At large heights, use the exact formula: at $h = R_E$, $g' = g/4$; at $h = 2R_E$, $g' = g/9$.

Q.7 Derive the variation of g with depth below the earth's surface. Show that g is maximum at the surface. (3 Marks)

Consider a point mass m at depth d below the surface (Fig. 7.8(b)). Its distance from the centre is $(R_E - d)$. The earth can be thought of as a smaller sphere of radius $(R_E - d)$ plus a spherical shell of thickness d .

- The outer shell exerts **no** force on m (shell theorem 2).
- The smaller sphere acts as if its mass M_s were at the centre.

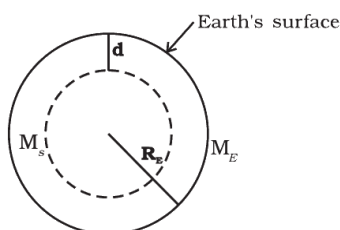


Fig. 7.8(b): g at a depth d . Only the smaller sphere of radius $(R_E - d)$ contributes to g

For uniform density, mass $\propto$ cube of radius:

$$\frac{M_s}{M_E} = \frac{(R_E - d)^3}{R_E^3}$$

$$F(d) = \frac{GM_s m}{(R_E - d)^2} = \frac{GM_E m (R_E - d)}{R_E^3}$$

$$g(d) = \frac{F(d)}{m} = \frac{GM_E}{R_E^2} \cdot \frac{R_E - d}{R_E} \Rightarrow g(d) = g \left(1 - \frac{d}{R_E} \right)$$

Conclusion: g decreases as we go down, and **becomes zero at the centre** ($d = R_E$). Since g decreases both above and below, **g is maximum on the earth's surface.**

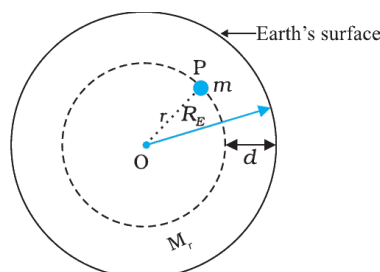


Fig. 7.7: A mass m in a mine at depth d . The earth is treated as spherically symmetric

Key Examination Points for Full Marks:

- Inside a uniform sphere, $F = \frac{GM_E m}{R_E^3} r \propto r$ — linear in r .
- Halfway to the centre, g becomes $g/2$.

Q.8 Define gravitational potential energy. Derive its expression at a distance r from the centre of the earth. (3 Marks)

Gravity is a **conservative** force, so a potential energy can be associated with it — the **gravitational potential energy**.

Near the earth's surface: Taking the force as constant (mg), the work done in lifting a particle from height h_1 to h_2 is $W_{12} = mg(h_2 - h_1)$. With $W(h) = mgh + W_0$: $W_{12} = W(h_2) - W(h_1)$.

At any distance r ($> R_E$): The force is $F = \frac{GM_E m}{r^2}$, directed towards the centre. The work done in lifting a particle from r_1 to r_2 is

$$W_{12} = \int_{r_1}^{r_2} \frac{GM_E m}{r^2} dr = -GM_E m \left(\frac{1}{r_2} - \frac{1}{r_1} \right)$$

So we associate the potential energy

$$W(r) = -\frac{GM_E m}{r} + W_1$$

Setting $W_1 = 0$ (zero potential energy at infinity):

$$V(r) = -\frac{GM_E m}{r}$$

The potential energy at a point is the work done in bringing the particle **from infinity** to that point. For two particles m_1 and m_2 at distance r : $V = -\frac{Gm_1 m_2}{r}$.

Gravitational potential is the potential energy per unit mass: $U = -\frac{GM}{r}$ (unit J kg^{-1}).

Key Examination Points for Full Marks:

- Only differences in PE are meaningful; mgh is an approximation valid for $h \ll R_E$.
- For a system of particles, the total PE is the sum over **all pairs** (superposition).
- **Example 7.3:** Four masses m at the corners of a square of side l : $W = -\frac{2Gm^2}{l} \left(2 + \frac{1}{\sqrt{2}} \right) = -5.41 \frac{Gm^2}{l}$
; potential at the centre $= -4\sqrt{2} \frac{Gm}{l}$.

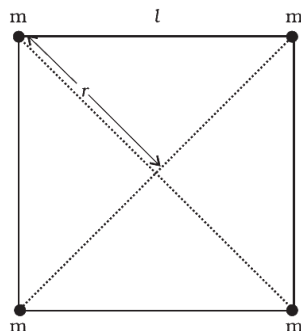


Fig. 7.9: Four equal masses at the corners of a square of side l

Q.9 What is escape speed? Derive its expression.

(3 Marks)

Escape speed: The minimum speed with which a body must be projected from the surface of a planet so that it escapes the planet's gravitational pull and never returns.

Suppose an object is thrown with speed V_i from a point at distance $(h + R_E)$ from the earth's centre, and reaches infinity with speed V_f . By conservation of energy:

$$\frac{1}{2}mV_i^2 - \frac{GmM_E}{h + R_E} = \frac{1}{2}mV_f^2$$

The right side is positive with minimum value zero. So the object reaches infinity if

$$\frac{1}{2}mV_i^2 - \frac{GmM_E}{h + R_E} \geq 0$$

The minimum value of V_i (escape speed) corresponds to equality. From the surface ($h = 0$):

$$v_e = \sqrt{\frac{2GM_E}{R_E}} = \sqrt{2gR_E}$$

Numerically, $v_e \approx 11.2$ km/s for the earth.

Key Examination Points for Full Marks:

- Escape speed does **not** depend on the mass of the body or the direction of projection (ignoring air).
- For the moon, $v_e \approx 2.3$ km/s — about five times smaller. Gas molecules can exceed this speed, which is why the **moon has no atmosphere**.
- **Example 7.4:** A projectile from a sphere M toward $4M$ (separation $6R$) must reach the neutral point N ($r = 2R$): $v = \left(\frac{3GM}{5R} \right)^{1/2}$.

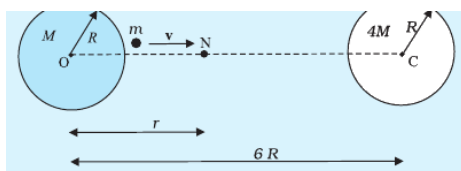


Fig. 7.10: Two spheres of masses M and $4M$ with centre separation $6R$; N is the neutral point

Q.10 Derive the orbital speed and time period of an earth satellite. Show (3 Marks) that $T^2 \propto (R_E + h)^3$.

Consider a satellite of mass m in a circular orbit at distance $(R_E + h)$ from the earth's centre with speed V . The gravitational force provides the centripetal force:

$$\frac{mV^2}{R_E + h} = \frac{GmM_E}{(R_E + h)^2}$$

$$V = \sqrt{\frac{GM_E}{R_E + h}}$$

V decreases as h increases. For $h = 0$: $V^2 = \frac{GM_E}{R_E} = gR_E$, so $V \approx 7.9 \text{ km/s}$.

Time period: The satellite covers $2\pi(R_E + h)$ in each orbit:

$$T = \frac{2\pi(R_E + h)}{V} = \frac{2\pi(R_E + h)^{3/2}}{\sqrt{GM_E}}$$

Squaring:

$$T^2 = k(R_E + h)^3, \quad k = \frac{4\pi^2}{GM_E}$$

This is **Kepler's law of periods** for earth satellites.

Satellite close to the surface: $T_0 = 2\pi\sqrt{\frac{R_E}{g}} = 2\pi\sqrt{\frac{6.4 \times 10^6}{9.8}} \text{ s} \approx 85 \text{ minutes}$.

Key Examination Points for Full Marks:

- The orbital speed and period do **not** depend on the mass of the satellite.
 - Relation with escape speed: $v_e = \sqrt{2} v_o$ (for an orbit close to the surface).
 - Example 7.5:** Mars' moon Phobos ($T = 7 \text{ h } 39 \text{ min}$, $R = 9.4 \times 10^3 \text{ km}$): $M_m = \frac{4\pi^2 R^3}{GT^2} = 6.48 \times 10^{23} \text{ kg}$.
- Martian year $= (1.52)^{3/2} \times 365 = 684 \text{ days}$.

Q.11 Derive the kinetic, potential and total energy of an orbiting satellite. Why is the total energy negative? (3 Marks)

Kinetic energy: Using $V^2 = \frac{GM_E}{R_E + h}$:

$$K = \frac{1}{2}mV^2 = \frac{GmM_E}{2(R_E + h)}$$

Potential energy (zero at infinity):

$$U = -\frac{GmM_E}{R_E + h}$$

Total energy:

$$E = K + U = -\frac{GmM_E}{2(R_E + h)}$$

- KE is positive; PE is negative. In magnitude, **KE is half of PE**: $K = -\frac{U}{2}$ and $E = -K = \frac{U}{2}$.
- The total energy is **negative**, because the satellite is a **bound system**. If the total energy were zero or positive, the object would escape to infinity. Satellites always remain at a finite distance from the earth, so their energy cannot be positive or zero.

Binding energy: The energy required to make the satellite escape: $\frac{GmM_E}{2(R_E + h)}$.

Key Examination Points for Full Marks:

- For elliptical orbits, KE and PE vary from point to point, but the total energy is constant and negative:

$$E = -\frac{GMm}{2a} \quad (a = \text{semi-major axis}).$$

- Example 7.8:** A 400 kg satellite moved from radius $2R_E$ to $4R_E$: $\Delta E = \frac{gmR_E}{8} = 3.13 \times 10^9 \text{ J}$;
 $\Delta K = -3.13 \times 10^9 \text{ J}$; $\Delta V = 6.25 \times 10^9 \text{ J}$.

Q.12 Why does an astronaut in an orbiting satellite feel weightless? (2 Marks)

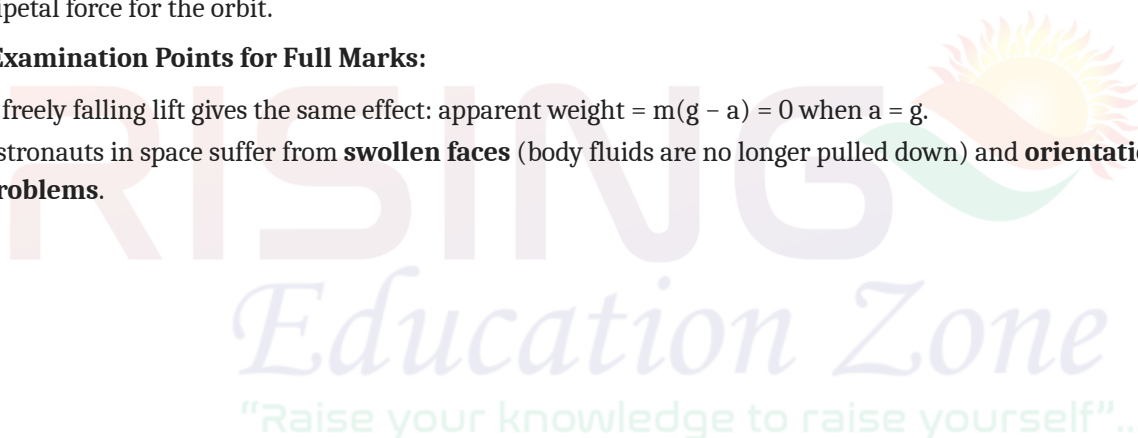
The weight of a body is the force with which the earth attracts it; we feel our weight through the **normal reaction** of the floor or support.

In an orbiting satellite, the satellite and the astronaut are both in **free fall** towards the earth with the same acceleration $g(h) = \frac{GM_E}{(R_E + h)^2}$. The floor of the satellite falls away just as fast as the astronaut, so it exerts **no normal reaction** on him. Hence he feels **weightless**.

This is **not** because gravity is small at that height — the gravitational force there is still large and provides the centripetal force for the orbit.

Key Examination Points for Full Marks:

- A freely falling lift gives the same effect: apparent weight = $m(g - a) = 0$ when $a = g$.
- Astronauts in space suffer from **swollen faces** (body fluids are no longer pulled down) and **orientation problems**.



Formula Sheet — Gravitation

Quantity	Formula	SI Unit
Kepler's 3rd law	$T^2 = \frac{4\pi^2}{GM} a^3$	—
Areal velocity	$\frac{dA}{dt} = \frac{L}{2m} = \text{constant}$	$\text{m}^2 \text{s}^{-1}$
Newton's law	$F = G \frac{m_1 m_2}{r^2}$	N
g at surface	$g = \frac{GM_E}{R_E^2} = \frac{4}{3}\pi G \rho R_E$	m s^{-2}
g at height h	$g(h) = \frac{GM_E}{(R_E + h)^2} \approx g \left(1 - \frac{2h}{R_E}\right)$	m s^{-2}
g at depth d	$g(d) = g \left(1 - \frac{d}{R_E}\right)$	m s^{-2}
g at latitude λ (rotation)	$g' = g - \omega^2 R_E \cos^2 \lambda$	m s^{-2}
Gravitational field	$E = \frac{GM}{r^2}$ (outside); $\frac{GM}{R^3}$ (inside solid sphere)	N kg^{-1}
Gravitational PE	$V = -\frac{GMm}{r}$	J
Gravitational potential	$U = -\frac{GM}{r}$	J kg^{-1}
Escape speed	$v_e = \sqrt{\frac{2GM}{R}} = \sqrt{2gR}$	m s^{-1}
Orbital speed	$v_o = \sqrt{\frac{GM}{r}}$; near surface $\sqrt{gR}$	m s^{-1}
Time period	$T = 2\pi\sqrt{\frac{r^3}{GM}}$; near surface $2\pi\sqrt{R/g}$	s
Energies in orbit	$K = \frac{GMm}{2r}, U = -\frac{GMm}{r},$ $E = -\frac{GMm}{2r}$	J

Constants: $G = 6.67 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$; $M_E \approx 6 \times 10^{24} \text{ kg}$; $R_E \approx 6400 \text{ km}$; $v_e = 11.2 \text{ km/s}$.

Dimensions: $G \left[\text{M}^{-1} \text{L}^3 \text{T}^{-2} \right]$; gravitational potential $\left[\text{L}^2 \text{T}^{-2} \right]$; gravitational intensity $\left[\text{L T}^{-2} \right]$.

NEET / JEE Quick Points and Tricks

- g varies:** at height $h = R, g/4$; $h = 2R, g/9$. At depth $R/2, g/2$; at the centre, 0. For the same g at height h and depth d (small values): $d = 2h$.
- Same density planets:** $g \propto R$ and $v_e \propto R$. **Same mass:** $g \propto 1/R^2, v_e \propto 1/\sqrt{R}$.
- Earth shrinks (mass same):** g at a point outside is unchanged; g on the new surface increases.
- Rotation:** g is minimum at the equator and maximum at the poles. Rotation does not affect g at the poles. If the earth rotated with $\omega = \sqrt{g/R}$ (≈ 17 times faster), g at the equator would be zero.
- Orbital vs escape speed:** $v_e = \sqrt{2} v_o$. To escape from an orbit, the extra speed needed is $(\sqrt{2} - 1) v_o$.
- Satellite energies:** $K:U:E = 1:-2:-1$. Energy to shift from r_1 to r_2 : $\frac{GMm}{2} \left(\frac{1}{r_1} - \frac{1}{r_2} \right)$.
- Orbit ratios:** $v \propto r^{-1/2}, T \propto r^{3/2}, L \propto r^{1/2}, K \propto 1/r$. For a 1% change in r, T changes by 1.5%.
- Geostationary satellite:** $T = 24 \text{ h}$, orbit radius $\approx 42,000 \text{ km}$ (height $\approx 36,000 \text{ km}$), above the equator, moving west to east. Polar satellites are low ($\approx 500\text{--}800 \text{ km}$).

9. **Neutral point** between masses m_1 and m_2 separated by d : distance from m_1 is $\frac{d\sqrt{m_1}}{\sqrt{m_1} + \sqrt{m_2}}$.
10. **Elliptical orbit**: $v_p r_p = v_a r_a$. Speed and KE are maximum at perihelion. The total energy depends only on the semi-major axis.
11. **Tunnel through the earth** (diameter or any chord): SHM with $T = 2\pi\sqrt{R/g} \approx 84.6$ min.
12. **Inside a shell**: field zero, potential constant ($-GM/R$). Inside a solid sphere: $V = -\frac{GM(3R^2 - r^2)}{2R^3}$; at the centre $V = -\frac{3GM}{2R}$.
13. **Weightlessness** in orbit is due to free fall, not absence of gravity. A pendulum clock does not work in a satellite ($T \rightarrow \infty$).
14. **Black hole condition**: escape speed $\geq c$, i.e. $\sqrt{2GM/R} \geq c$ (Schwarzschild radius $R = 2GM/c^2$).

